MÖBIUS AND DELTA TRANSFORMS IN THE UNIFICATION OF CONTINUOUS-DISCRETE SPACES

T. BAKHTIAR[1], SAMSURIZAL[2], N. ALIATININGTYAS[3]

Abstract

It is well-known that in control theory the stability region of continuous-time system is laid in the left half plane of complex space, while that of discrete-time system is dwelled inside a unit circle. The former fact might be shown by exploiting the Laplace transform and the later by utilizing the corresponding zeta transform. In this paper we revealed the connectivity of both regions by employing Möbius transform. We also used the same transform to derive continuous/discrete-time counterpart of several existing results, including Bode integral and Poisson-Jensen formula. We then demonstrated their unification property by using delta transform. Some numerical examples were provided to verify our results.

Keywords: continuous-discrete unification, Möbius transform, delta transform, stability

1 INTRODUCTION

In control theory, some results for continuous-time and discrete-time such as regions of stability, expressions for minimum tracking error and energy regulation are derived independently. It is well-known that the stability region of continuous-time system is laid in the left hand side of imaginary axis in complex space, while that of discrete-time system is located inside a unit circle. In this paper, instead of using rigorous derivation we aim to utilize the strength of the Möbius transform to obtain the counterpart of existing results, which cover the Bode integral and Poisson-Jensen formula.

While discrete system is obtained from continuous-time system by sampling, expressions in continuous and discrete domains are not quite clear. This is because the underlying continuous domain descriptions cannot be obtained by setting sampling period to zero in the discrete domain approximations. In optimal tracking error control problem, for instance, contribution of non-minimum phase zeros for continuous and discrete-time systems are provided in completely different ways, see [3, 7]. The delta operator has often been

[1]Department of Mathematics, Bogor Agricultural University, Bogor.
[2]Madrasah Aliyah Negeri 1, Bandar Lampung, Indonesia.
[3]Department of Mathematics, Bogor Agricultural University, Bogor.

proven entailing many advantages in connecting discrete-time and continuous-time systems, such as control synthesis [4], control design [5], estimation [9] and filtering [11]. We then provide the delta domain version of some existing results.

2 REGION OF STABILITY

The most commonly used definitions of stability are based on the magnitude of the system response in the steady state. A system is said to be (asymptotically) stable if its response to any initial conditions decays to zero asymptotically in the steady state, otherwise is said to be unstable. From the perspective of the forced response of the system for a bounded input, a system is said to be bounded-input and bounded-output (BIBO) stable if its response to any bounded input remains bounded. This section reviews the region of stability for continuous-time and discrete-time systems.

2.1 Continuous-time System

Consider a linear continuous-time system given by an initial value problem of n-th order differential equation:

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{j=0}^{m} b_j u^{(j)}(t),\ y^{(i)}(0) = u^{(j)}(0) = 0, \tag{1}$$

where $y(t)$ and $u(t)$ respectively are the output and input of the system, $a_i\,(i = 0, \dots, n)$ and $b_j\ (j = 0, \dots, m)$ are real coefficients, and a_n and b_m are non-zeros. Note that in (1), $y^{(i)}$ denotes the i-th derivative of y with respect to time t. System (1) can be expressed in frequency domain, i.e., s-domain, by Laplace transform

$$\sum_{i=0}^{n} a_i s^i Y(s) = \sum_{j=0}^{m} b_j s^j U(s),$$

where $Y(s) := \mathcal{L}\{y(t)\}$ and $U(s) := \mathcal{L}\{u(t)\}$ respectively denote the Laplace transform of $y(t)$ and $u(t)$. The system possesses the following transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}, \tag{2}$$

where $H(s)$ represents the transfer function between input $U(s)$ and output $Y(s)$, i.e., $Y(s) = H(s)U(s)$. If it is assumed that H has no zeros and poles in the same location, then we may write (2) as

$$H(s) = \frac{c(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}, \tag{3}$$

where z_j $(j = 1, \ldots, m)$ and p_i $(i = 1, \ldots, n)$ are zeros and poles of H, respectively, and c is a real constant. As we assume that $n \geq m$, (3) can be written in term of partial fraction decomposition as follows,

$$H(s) = \sum_{i=1}^{n} \frac{c_i}{s - p_i}, \tag{4}$$

where $c_i = \lim_{s \to p_i} (s - p_i) H(s)$, $i = 1, \ldots, n$.

By considering an impulse input function, i.e., $U(s) = 1$ or equivalently $u(t) = \delta(t)$, then the time domain solution for (1) can be obtained by applying the inverse Laplace transform:

$$y(t) = \sum_{i=1}^{n} \mathcal{L}^{-1} \left\{ \frac{c_i}{s - p_i} \right\} = \sum_{i=1}^{n} c_i e^{p_i t}. \tag{5}$$

It can easily be seen from (5) that system (1) is (asymptotically) stable, i.e., $\lim_{t \to \infty} y(t) = 0$, if and only if $\operatorname{Re} p_i < 0$ for all $i = 1, \ldots, n$. In other words, the stability region of continuous-time system is laid in the left half plane of complex space.

2.2 Discrete-time System

Consider a linear discrete-time system represented by a single n-th order difference equation relating the output y to the input u:

$$\sum_{i=0}^{n} a_i y(k + i) = \sum_{j=0}^{m} b_j u(k + j), \tag{6}$$

for $k = 0, 1, \ldots$ and $y(i) = u(j) = 0$. Zeta transform provides the z-domain version of system (6), i.e.,

$$\sum_{i=0}^{n} a_i z^i Y(z) = \sum_{j=0}^{m} b_j z^j U(z),$$

where $Y(z) := \mathcal{Z}\{y(k)\}$ and $U(z) := \mathcal{Z}\{u(k)\}$ respectively denote the zeta transform of $y(k)$ and $u(k)$. Thus we have the following transfer function

$$H(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0}. \tag{7}$$

Note that if all poles of $H(z)$ lie inside the unit circle $\{z : |z| < 1\}$, (7) can be rewritten as a power series

$$H(z) = \sum_{i=0}^{\infty} c_i z^{-i}. \tag{8}$$

By applying the inverse zeta transform we have

$$y(k) = \sum_{i=0}^{k} c_i u(k-i). \tag{9}$$

Let $|u(i)| \le M$ for all i and $C = \sum_{i=0}^{\infty} c_i$. Then,

$$|y(k)| = \left| \sum_{i=0}^{k} c_i u(k-i) \right| \le \sum_{i=0}^{k} |c_i||u(k-i)| \le MC$$

for each $k = 0, 1, \ldots$. Therefore the linear system (6) is stable. The proof of the converse can be found in [6]. As we completed the proof, it is shown that the stability region of a linear discrete-time system lies inside the unit circle.

3 MÖBIUS TRANSFORM

It has been revealed in the previous section that the region of stability of continuous-time system is the left half plane of complex space, while that of discrete-time system is the interior of a unit circle. In this section we will utilize the Möbius transform [2] to show the interconnection between these two regions. In particular, we demonstrate that the region of stability of discrete domain can be asserted by Möbius transforming that of continuous domain.

Definition 1 (Möbius Transform)**.** *Transformation*

$$s = \mathcal{M}(z) := \frac{az+b}{cz+d}, \quad ad - bc \neq 0, \tag{10}$$

where a, b, c and d are complex-valued constants, is called Möbius transform from variable z to variable s.

When $c \neq 0$, equation (10) can be written

$$s = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d},$$

from which we can see that condition $ad - bc \neq 0$ ensures that we do not have a constant function. If we assign $\mathcal{M}(\infty) = \infty$ for $c = 0$, and $\mathcal{M}(\infty) = \frac{a}{c}$ and $\mathcal{M}(-\frac{d}{c}) = \infty$ for $c \neq 0$, then Möbius transform (10) is certainly a bijective mapping of the extended z-domain onto the extended s-domain. When a given point s is the image of some point z under transformation (10), the point z is retrieved by its inverse

$$z = \mathcal{M}^{-1}(s) := \frac{-ds+b}{cs-a}, \quad ad - bc \neq 0. \tag{11}$$

We can verify that the inverse function (11) is also a bijective mapping by using the definition $\mathcal{M}^{-1}(\infty) = \infty$ for $c = 0$, and $\mathcal{M}^{-1}(\frac{a}{c}) = \infty$ and $\mathcal{M}^{-1}(\infty) = -\frac{d}{c}$ for $c \neq 0$.

3.1 Region of Stability Mapping

In Section 2 the regions of stability for continuous-time and discrete-time systems were derived separately by employing the Laplace and zeta transforms, respectively. In this part we will show that the region of stability for discrete-time system can be obtained by Möbius transforming that for continuous-time system. To facilitate our analysis, for $\mathbb{C}$ the complex space we define the following sets: $\mathbb{C}^- := \{s \in \mathbb{C} : \operatorname{Re} s < 0\}$, $\mathbb{C}^+ := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^c := \{z \in \mathbb{C} : |z| \geq 1\}$ and $\bar{\mathbb{D}}^c := \{z \in \mathbb{C} : |z| > 1\}$.

For the analysis we consider a special case of Möbius transform where $a = c = d = 1$ and $b = -1$, that is

$$s = \frac{z-1}{z+1}. \tag{12}$$

Let $s = \alpha + j\beta$, $j = \sqrt{-1}$, is an arbitrary point in $\mathbb{C}^-$, that is $\alpha < 0$. Then we have

$$2\alpha < -2\alpha \Leftrightarrow (\alpha+1)^2 + \beta^2 < (\alpha-1)^2 + \beta^2.$$

The above is nothing but the modulo of complex numbers

$$|(\alpha+1) + j\beta| < |(\alpha-1) + j\beta| \Leftrightarrow |s+1| < |s-1|.$$

By applying (12) to above inequality we have

$$\begin{aligned} \left|\frac{z-1}{z+1} + 1\right| < \left|\frac{z-1}{z+1} - 1\right| \quad &\Leftrightarrow \quad \left|\frac{2z}{z+1}\right| < \left|\frac{-2}{z+1}\right| \\ &\Leftrightarrow \quad |z| < 1. \end{aligned}$$

It is shown that $s \in \mathbb{C}^-$ corresponds to $z \in \mathbb{D}$, which reveals the interconnection between region of stability in s-domain and its counterpart in z-domain.

3.2 Other Continuous-Discrete Relationships

Now we further exploit the use of special Möbius transform (12) to unveil the counterpart of existing continuous-time or discrete-time results, which covers Bode integral and Poisson-Jensen formula.

Theorem 2 (Bode integral in s-domain)**.** *Let $g(s)$ be an analytic function in $\mathbb{C}^+$. Denote that $g(j\omega) = g_1(\omega) + jg_2(\omega)$ and $g(s) = \overline{g(\bar{s})}$, i.e., g is conjugate symmetric. Then,*

$$g'(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g_1(\omega) - g_1(0)}{\omega^2}\, d\omega. \tag{13}$$

Proof. See [10, pp. 53].

The discrete-time counterpart of Bode integral is given as follows.

Theorem 3 (Bode integral in z-domain)**.** *Let $f(z)$ be an analytic function in $\mathbb{D}^c$. Denote that $f(e^{j\theta}) = f_1(\theta) + jf_2(\theta)$ and $f(z) = \overline{f(\bar{z})}$, i.e., f is conjugate symmetric. Then,*

$$2f'(1) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f_1(\theta) - f_1(0)}{1 - \cos\theta}\, d\theta. \tag{14}$$

PROOF. From (12) we have the following mappings: $\{s = 0\} \mapsto \{z = 1\}$ and $\{s = j\omega\} \mapsto \{z = e^{j\theta}\}$, where $\omega = \tan\frac{1}{2}\theta$. Thus we also have $d\omega = \frac{1}{1+\cos\theta}\, d\theta$ and $\{\omega = \pm\infty\} \mapsto \{\theta = \pm\pi\}$. Since $f(z) = g(\frac{z-1}{z+1})$ we obtain $g'(0) = 2f'(1)$. Equation (14) is claimed from (13) by substitution.

Theorem 4 (Poisson-Jensen formula in z-domain)**.** *Let f is analytic in $\bar{\mathbb{D}}^c$ and d_i $(i = 1, \ldots, n_d)$ be the zeros of f in $\bar{\mathbb{D}}^c$, counting their multiplicities. If $z \in \bar{\mathbb{D}}^c$ and $f(z) \neq 0$, then*

$$\begin{aligned}\log|f(z)| &= \frac{1}{\pi}\int_0^{\pi} \mathrm{Re}\left[\frac{ze^{j\theta}+1}{ze^{j\theta}-1}\right] \log|f(e^{j\theta})|\, d\theta \\ &\quad - \sum_{i=1}^{n_d} \log\left|\frac{1 - \bar{d}_i z}{z - d_i}\right|. \end{aligned}\tag{15}$$

PROOF. The Poisson-Jensen formula can be found in many standard books on complex analysis. See for instance [1].

The continuous-time counterpart of Poisson-Jensen formula is provided as follows.

Theorem 5 (Poisson-Jensen formula in s-domain)**.** *Let g is analytic in $\mathbb{C}^+$ and c_i $(i = 1, \ldots, n_c)$ be the zeros of g in $\mathbb{C}^+$, counting their multiplicities. If $s \in \mathbb{C}^+$ and $g(s) \neq 0$, then*

$$\begin{aligned}\log|g(s)| &= \frac{2}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{1+j\omega s}{s+j\omega}\right] \frac{\log|g(j\omega)|}{1+\omega^2}\, d\omega \\ &\quad - \sum_{i=1}^{n_c} \log\left|\frac{\bar{c}_i + s}{c_i - s}\right|. \end{aligned}\tag{16}$$

PROOF. Perform transformation (12) over Theorem 4 to prove it.

4 DELTA TRANSFORM

A book that provides a comprehensive account on delta operator is [8]. The delta operator δ is define as the following forward difference

$$\delta \triangleq \frac{q-1}{T},$$

where q is the forward shift operator commonly used in discrete-time case and $T > 0$ is the sampling time. For any sequence $x(k)$, $k = 1, 2, \ldots$, delta operator gives

$$\delta x(k) = \frac{(q-1)x(k)}{T} = \frac{x(k+1) - x(k)}{T}.$$

By taking the zeta transform of above equation we obtain

$$\delta X(z) = \frac{z-1}{T} X(z).$$

We may say that in z-plane the delta operator will translate a point $z \in \mathbb{C}$ one unit to the left and then scale it by factor of $\frac{1}{T}$. Later, the variable δ is used as the delta operator variable and is analogous to the Laplace variable s for continuous-time systems and the zeta transform variable z for discrete-time systems. We then obtain the relationship between variable z and variable δ as follows,

$$\delta = \frac{z-1}{T} \Leftrightarrow z = T\delta + 1. \tag{17}$$

For any sequence $x(k)$ we define its delta transform by

$$\mathcal{D}\{x(k)\} = X_T(\delta) := T \sum_{k=0}^{\infty} x(k)(T\delta + 1)^{-k}.$$

Now we ready to present the δ-domain counterparts of previous theorems. For time sampling T we define the following sets: $\mathbb{D}_T := \{\delta \in \mathbb{C} : |T\delta + 1| < 1\}$, $\mathbb{D}_T^{\mathrm{c}} := \{\delta \in \mathbb{C} : |T\delta + 1| \geq 1\}$ and $\bar{\mathbb{D}}_T^{\mathrm{c}} := \{\delta \in \mathbb{C} : |T\delta + 1| > 1\}$.

Theorem 6 (Bode integral in δ-domain)**.** *Let h be an analytic function in $\mathbb{D}_T^{\mathrm{c}}$. Denote that $h(e^{(j\omega T-1)/T}) = h_1(\omega T) + jh_2(\omega T)$ and $h(\delta) = \overline{h(\bar{\delta})}$, i.e., h is conjugate symmetric. Then,*

$$\frac{2h'(0)}{T} = \frac{T}{\pi} \int_{-\pi/T}^{\pi/T} \frac{h_1(\omega T) - h_1(0)}{1 - \cos \omega T} \, d\omega. \tag{18}$$

Proof. Consider the relationship $h(\delta) = f(T\delta + 1)$, where f is defined in Theorem 3. Since f is analytic in $\mathbb{D}^{\mathrm{c}}$, then h is analytic in $\mathbb{D}_T^{\mathrm{c}}$. From (17) we can easily verify that $h'(0) = Tf'(1)$, $d\theta = T\,d\omega$ and $\{z = e^{j\theta}\} \mapsto \{\delta = e^{(j\omega T-1)/T}\}$.

Theorem 7 (Poisson-Jensen formula in δ-domain)**.** *Let h is analytic in $\bar{\mathbb{D}}_T^{\mathrm{c}}$ and σ_i $(i = 1, \ldots, n_\sigma)$ be the zeros of h in $\bar{\mathbb{D}}_T^{\mathrm{c}}$, counting their multiplicities. If $\delta \in \bar{\mathbb{D}}_T^{\mathrm{c}}$ and $h(\delta) \neq 0$, then*

$$\begin{aligned} \log|h(\delta)| &= \\ & \frac{T}{\pi} \int_0^{\pi/T} \operatorname{Re}\left[\frac{(T\delta+1)e^{j\omega T} + 1}{(T\delta+1)e^{j\omega T} - 1}\right] \log|h(e^{(j\omega T-1)/T})| \, d\omega \\ & - \sum_{i=1}^{n_\sigma} \log\left|\frac{T\delta\bar{\sigma}_i + \bar{\sigma}_i + \delta}{\delta - \sigma_i}\right|. \end{aligned} \tag{19}$$

5 NUMERICAL EXAMPLE

In this section we provide a simple illustrative example to verify our result and to show the unification property between δ-domain and s-domain results.

5.1 Verification

To verify the result presented in Theorem 3 we consider a function f defined in z-domain as follows,

$$f(z) = \frac{3z+1}{z-p}, \quad -1 < p < 1. \tag{20}$$

Since pole p lies inside the unit circle, indeed f is analytic in $\mathbb{D}^c$ and it can be written as $f(e^{j\theta}) = f_1(\theta) + jf_2(\theta)$, where

$$\begin{aligned} f_1(\theta) &= \frac{3-p+(1-3p)\cos\theta}{p^2-2p\cos\theta+1}, \\ f_2(\theta) &= -\frac{(1+3p)\sin\theta}{p^2-2p\cos\theta+1}. \end{aligned}$$

Further, the LHS of (14) is given by

$$2f'(1) = -\frac{2(3p+1)}{(p-1)^2}. \tag{21}$$

Whereas, for the RHS we have

$$\begin{aligned} &\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f_1(\theta)-f_1(0)}{1-\cos\theta}\, d\theta \\ &\quad = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\frac{3-p+(1-3p)\cos\theta}{p^2-2p\cos\theta+1} - \frac{4}{1-p}}{1-\cos\theta}\, d\theta \\ &\quad = \frac{(p+1)(3p+1)}{\pi(p-1)} \int_{-\pi}^{\pi} \frac{d\theta}{p^2-2p\cos\theta+1}. \end{aligned} \tag{22}$$

Figure 1 shows that numerical calculation of LHS in (21) and that of RHS in (22) for varying stable pole p are coinciding.

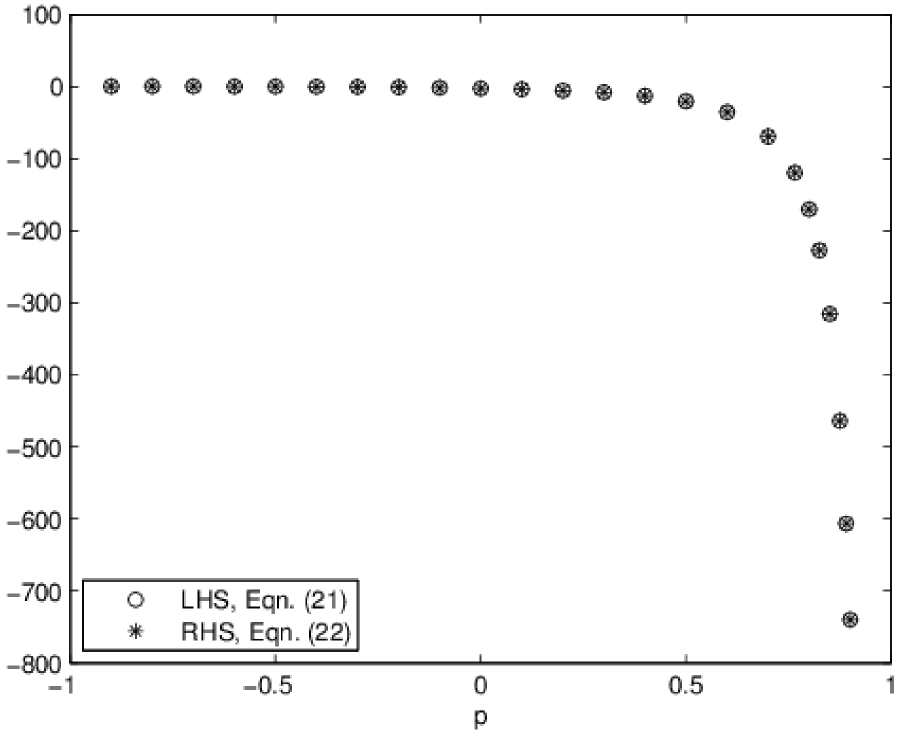

Figure 1: Bode integral in z-domain

5.2 Unification

Since $g(s) = f(\frac{1+s}{1-s})$, from (20) we obtain

$$g(s) = \frac{4 + 2s}{(1 + p)s + 1 - p}, \tag{23}$$

from which we also have

$$g'(0) = -\frac{2(3p + 1)}{(1 - p)^2}.$$

In the δ-domain, from the relationship $h(\delta) = f(T\delta + 1)$ we have

$$h(\delta) = \frac{3T\delta + 4}{T\delta + 1 - p}, \tag{24}$$

and thus

$$h'(0) = -\frac{3Tp + T}{(1 - p)^2}.$$

Therefore, the unification property of Theorems 2, 3 and 6 is unveiled by fact that

$$\lim_{T\to 0}\frac{2h'(0)}{T}=\lim_{T\to 0}-\frac{3Tp+T}{T(1-p)^2}=-\frac{3p+1}{(1-p)^2}=g'(0),$$

which show that the LHS of (18) converges to that of (13) as the sampling time T decreases.

6 CONCLUDING REMARK

It has been shown that Möbius transform can be utilized to derive counterpart result without involving any rigorous derivation. The delta transform, which takes a time sampling into account, can be used to show the unification property between continuous and discrete results. The approach describes in this paper can then be exploited to derive many more counterpart expressions.

References

[1] Ash RB, Novinger WP. 2007. *Complex Variables.* 2nd Ed. Dover.

[2] Brown JW, Churchill RV. 2009. *Complex Variables and Application.* 8th Ed. McGraw-Hill.

[3] Chen G, Chen J, Middleton R. 2002. Optimal tracking performance for SIMO systems. *IEEE T. Automat. Contr.* 47(10):1770–1775.

[4] Collins EG. 1999. A Delta Operator Approach to Discrete-time $\mathcal{H}_\infty$ Control. *Int. J. Control.* 72(4):315–320.

[5] Emami T. 2007. A Unified Procedure for Continuous-Time and Discrete-time Root-Locus and Bode Design. *Proc. Amer. Contr. Conf.* pp 2509–2514.

[6] Fisher SD. 1999. *Complex Variables.* 2nd Ed. Dover.

[7] Hara S, Bakhtiar T, Kanno M. 2007. The Best Achievable $\mathcal{H}_2$ Tracking Performances for SIMO Feedback Control Systems. *J. Contr. Sci. Eng.* pp 1–12.

[8] Middleton RH, Goodwin GC. 1990. *Digital Control and Estimation: A Unified Approach.* Prentice Hall.

[9] Ninness BM, Goodwin GC. 1991. The Relationship Between Discrete Time and Continuous Time Linear Estimation," in Sinha, N.K. and Rao, G.P. (Eds). *Identification of Continuous-Time Systems, International Series on Microprocessor-Based Systems Engineering.* 7:79–122.

[10] Seron MM, Braslavsky JH, Goodwin GC. 1997. *Fundamental Limitations in Filtering and Control.* Springer.

[11] Yang H, Xia Y, Shi P, Fu M. 2011. A Novel Delta Operator Kalman Filter Design and Convergence Analysis. *IEEE T. Cir. Sys. Part 1: Regular Papers.* 58(10):2458–2468.